SOMMES TÉLESCOPİQUES

Exercice 1 :

1. $\frac{k-5}{k(k^2-1)} = \frac{a}{k-1} + \frac{b}{k} + \frac{c}{k+1}$

On multiplie par $(k-1)$: $\frac{k-5}{k(k+1)} = a + \frac{b(k-1)}{k} + \frac{c(k-1)}{k+1}$

$\frac{1-5}{1(1+1)} = a$ avec $k = 1$

$\frac{-4}{2} = a = -2$

On multiplie par k : $\frac{k-5}{k^2-1} = \frac{ak}{k-1} + b + \frac{ck}{k+1}$

$\frac{0-5}{0^2-1} = b$ avec $k = 0$

$5 = b$ avec $k = 0$

On multiplie par $(k+1)$: $\frac{k-5}{k(k-1)} = \frac{a(k+1)}{k-1} + \frac{b(k+1)}{k} + c$

$\frac{-1-5}{-1(-1-1)} = c$ avec $k = -1$

$\frac{-6}{2} = c = -3$

Donc $\frac{k-5}{k(k^2-1)} = \frac{-2}{k-1} + \frac{5}{k} + \frac{-3}{k+1}$

2. $S_n = \sum_{k=2}^{n} \frac{k-5}{k(k^2-1)}$

$= \sum_{k=2}^{n} \frac{-2}{k-1} + \frac{5}{k} + \frac{-3}{k+1}$

$= -2\sum_{k=2}^{n} \frac{1}{k-1} + 5\sum_{k=2}^{n} \frac{1}{k} - 3\sum_{k=2}^{n} \frac{1}{k+1}$

$= -2\sum_{k=1}^{n-1} \frac{1}{k} + 5\sum_{k=2}^{n} \frac{1}{k} - 3\sum_{k=3}^{n+1} \frac{1}{k}$

$= -2\left(\frac{1}{1} + \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{n}\right) + 5\sum_{k=2}^{n} \frac{1}{k} - 3\left(-\frac{1}{2} + \sum_{k=2}^{n} \frac{1}{k} + \frac{1}{n+1}\right)$

$= -2 - 2\sum_{k=2}^{n} \frac{1}{k} + \frac{2}{n} + 5\sum_{k=2}^{n} \frac{1}{k} + \frac{3}{2} - 3\sum_{k=2}^{n} \frac{1}{k} - \frac{3}{n+1}$

$= -\frac{1}{2} + \frac{2}{n} - \frac{3}{n+1}$

$= \frac{-n(n+1) + 4(n+1) - 6n}{2n(n+1)}$

$= \frac{-n^2 - n + 4n + 4 - 6n}{2n(n+1)}$

$= \frac{-n^2 - 3n + 4}{2n(n+1)}$

3. $\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} -\frac{1}{2} + \frac{2}{n} - \frac{3}{n+1} = -\frac{1}{2}$

Exercice 2 :

1. $\frac{1}{(x+1)(x+2)(x+3)} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{x+3}$

On multiplie par $(x+1)$: $\frac{1}{(x+2)(x+3)} = a + \frac{b(x+1)}{x+2} + \frac{c(x+1)}{x+3}$

$\frac{1}{(-1+2)(-1+3)} = a$ avec $x = -1$

$\frac{1}{1\times 2} = a = \frac{1}{2}$

On multiplie par $(x+2)$: $\frac{1}{(x+1)(x+3)} = \frac{a(x+2)}{x+1} + b + \frac{c(x+2)}{x+3}$

$\frac{1}{(-2+1)(-2+3)} = b$ avec $x = -2$

$\frac{1}{-1\times 1} = b = -1$

On multiplie par $(x+3)$: $\frac{1}{(x+1)(x+2)} = \frac{a(x+3)}{x+1} + \frac{b(x+3)}{x+2} + c$

$\frac{1}{(-3+1)(-3+2)} = c$ avec $x = -3$

$\frac{1}{-2\times(-1)} = c = \frac{1}{2}$

Donc $\frac{1}{(x+1)(x+2)(x+3)} = \frac{\frac{1}{2}}{x+1} + \frac{-1}{x+2} + \frac{\frac{1}{2}}{x+3}$

2. $S_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)}$

$= \sum_{k=0}^{n} \frac{\frac{1}{2}}{k+1} + \frac{-1}{k+2} + \frac{\frac{1}{2}}{k+3}$

$= \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k+1} - \sum_{k=0}^{n} \frac{1}{k+2} + \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k+3}$

$= \frac{1}{2}\sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=2}^{n+2} \frac{1}{k} + \frac{1}{2}\sum_{k=3}^{n+3} \frac{1}{k}$

$= \frac{1}{2}\left(\sum_{k=3}^{n+1} \frac{1}{k} + 1 + \frac{1}{2}\right) - \left(\sum_{k=3}^{n+1} \frac{1}{k} + \frac{1}{2} + \frac{1}{n+2}\right) + \frac{1}{2}\left(\sum_{k=3}^{n+1} \frac{1}{k} + \frac{1}{n+2} + \frac{1}{n+3}\right)$

$$= \frac{1}{2}\sum_{k=3}^{n+1}\frac{1}{k} + \frac{1}{2} + \frac{1}{4} - \sum_{k=3}^{n+1}\frac{1}{k} - \frac{1}{2} - \frac{1}{n+2} + \frac{1}{2}\sum_{k=3}^{n+1}\frac{1}{k} + \frac{1}{2}\times\frac{1}{n+2} + \frac{1}{2}\times\frac{1}{n+3}$$
$$= \frac{1}{4} - \frac{1}{2}\times\frac{1}{n+2} + \frac{1}{2}\times\frac{1}{n+3}$$
$$= \frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{4(n+2)(n+3)}$$
$$= \frac{n^2+3n+2n+6-2n-6+2n+4}{4(n+2)(n+3)}$$
$$= \frac{n^2+5n+4}{4(n+2)(n+3)}$$

3. $\lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to+\infty} \frac{1}{4} - \frac{1}{2}\times\frac{1}{n+2} + \frac{1}{2}\times\frac{1}{n+3} = \frac{1}{4}$

<u>Exercice 3</u> :

- $S_1 = \sum_{k=1}^{n}\frac{k}{(k+1)!}$
$$= \sum_{k=1}^{n}\frac{k+1-1}{(k+1)!}$$
$$= \sum_{k=1}^{n}\frac{k+1}{(k+1)!} - \sum_{k=1}^{n}\frac{1}{(k+1)!}$$
$$= \sum_{k=1}^{n}\frac{1}{k!} - \sum_{k=2}^{n+1}\frac{1}{k!}$$
$$= \sum_{k=2}^{n}\frac{1}{k!} + \frac{1}{1!} - \left(\sum_{k=2}^{n}\frac{1}{k!} + \frac{1}{(n+1)!}\right)$$
$$= 1 - \frac{1}{(n+1)!}$$

- $S_2 = \sum_{k=1}^{n}\frac{1}{\sqrt{k}+\sqrt{k+2}}$
$$= \sum_{k=1}^{n}\frac{\sqrt{k}-\sqrt{k+2}}{(\sqrt{k}+\sqrt{k+2})(\sqrt{k}-\sqrt{k+2})}$$
$$= \sum_{k=1}^{n}\frac{\sqrt{k}-\sqrt{k+2}}{k-(k+2)}$$
$$= \sum_{k=1}^{n}\frac{\sqrt{k}-\sqrt{k+2}}{-2}$$
$$= \frac{1}{2}\sum_{k=1}^{n}\sqrt{k+2}-\sqrt{k}$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}\sqrt{k+2} - \sum_{k=1}^{n}\sqrt{k}\right)$$
$$= \frac{1}{2}\left(\sum_{k=3}^{n+2}\sqrt{k} - \sum_{k=1}^{n}\sqrt{k}\right)$$
$$= \frac{1}{2}\left(\sum_{k=3}^{n+2}\sqrt{k} - \left(\sum_{k=3}^{n+2}\sqrt{k} + \sqrt{1} + \sqrt{2} - \sqrt{n+1} - \sqrt{n+2}\right)\right)$$
$$= \frac{1}{2}\left(-1 - \sqrt{2} + \sqrt{n+1} + \sqrt{n+2}\right)$$

- $S_3 = \sum_{k=1}^{n}\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}$
$$= \sum_{k=1}^{n}\frac{\sqrt{2k-1}-\sqrt{2k+1}}{(\sqrt{2k-1}+\sqrt{2k+1})(\sqrt{2k-1}-\sqrt{2k+1})}$$
$$= \sum_{k=1}^{n}\frac{\sqrt{2k-1}-\sqrt{2k+1}}{2k-1-(2k+1)}$$
$$= \sum_{k=1}^{n}\frac{\sqrt{2k-1}-\sqrt{2k+1}}{-2}$$
$$= \frac{1}{2}\sum_{k=1}^{n}\sqrt{2k+1}-\sqrt{2k-1}$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}\sqrt{2k+1} - \sum_{k=1}^{n}\sqrt{2k-1}\right)$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}\sqrt{2k+1} - \sum_{j=0}^{n-1}\sqrt{2(j+1)-1}\right)$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}\sqrt{2k+1} - \sum_{k=0}^{n-1}\sqrt{2k+1}\right)$$
$$= \frac{1}{2}\left(\sqrt{2n+1} - \sqrt{2\times 0+1}\right)$$
$$= \frac{1}{2}\left(\sqrt{2n+1} - 1\right)$$

- $S_4 = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{n+1-k}\right)$
$$= \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{n+1-k}$$
$$= \sum_{k=1}^{n}\frac{1}{k} - \sum_{i=1}^{n}\frac{1}{i} = 0$$

Exercice 4 :

$$xS_n(x) - S_n(x) = x\sum_{k=0}^{n} kx^k - \sum_{k=0}^{n} kx^k$$
$$= \sum_{k=0}^{n} kx^{k+1} - \sum_{k=0}^{n} kx^k$$
$$= \sum_{k=1}^{n+1} (k-1)x^k - \left(\sum_{k=1}^{n+1} kx^k - (n+1)x^{n+1}\right)$$
$$= \sum_{k=1}^{n+1} (k-1)x^k - \sum_{k=1}^{n+1} kx^k + (n+1)x^{n+1}$$
$$= \sum_{k=1}^{n+1} \left((k-1)x^k - kx^k\right) + (n+1)x^{n+1}$$
$$= \sum_{k=1}^{n+1} (kx^k - x^k - kx^k) + (n+1)x^{n+1}$$
$$= -\sum_{k=1}^{n+1} x^k + (n+1)x^{n+1}$$
$$= -x\frac{1-x^{n+1}}{1-x} + (n+1)x^{n+1}$$
$$= x\frac{-(1-x^{n+1}) + (n+1)x^n(1-x)}{1-x}$$
$$= x\frac{-1+x^{n+1}+(n+1)x^n-(n+1)x^{n+1}}{1-x}$$
$$= x\frac{x^{n+1}(1-n-1)+(n+1)x^n-1}{1-x}$$
$$= x\frac{-nx^{n+1}+(n+1)x^n-1}{1-x}$$

Or, $xS_n(x) - S_n(x) = (x-1)S_n(x)$

D'où $S_n(x) = x \times \dfrac{-nx^{n+1}+(n+1)x^n-1}{(1-x)(x-1)}$

$$= x \times \frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$$

Autre méthode : $\left(\sum_{k=0}^{n} x^k\right)' = \sum_{k=0}^{n} (x^k)' = \sum_{k=0}^{n} kx^{k-1}$

Donc $S_n(x) = \sum_{k=0}^{n} kx^k = x\sum_{k=0}^{n} kx^{k-1} = x\left(\sum_{k=0}^{n} x^k\right)'$

Mais $\left(\sum_{k=0}^{n} x^k\right)' = \left(\dfrac{1-x^{n+1}}{1-x}\right)' = \dfrac{-(n+1)x^n(1-x)-(-1)(1-x^{n+1})}{(1-x)^2}$

$$= \frac{-(n+1)x^n + (n+1)x^{n+1} + 1 - x^{n+1}}{(1-x)^2}$$

$$= \frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$$

Et donc $S_n(x) = x\dfrac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$

Exercice 5 : *Formule de König-Huyghens*

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2x_i\overline{x} + \overline{x}^2$$
$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\overline{x}^2\right)$$
$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\overline{x}n\overline{x} + n\overline{x}^2\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\overline{x}^2 + \overline{x}^2$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{x}^2 \quad \Longrightarrow \quad V(X) = E(X^2) - \left(E(X)\right)^2$$

Exercice 6 :

1° $k^2(k+1)^2 + (k+1)^2 + k^2 = k^2(k^2+2k+1) + k^2 + 2k + 1 + k^2$
$= k^4 + 2k^3 + k^2 + k^2 + 2k + 1 + k^2$
$= k^4 + 2k^3 + 3k^2 + 2k + 1$

Et $(k^2+k+1)^2 = k^4 + k^2 + 1 + 2k^3 + 2k^2 + 2k$
$= k^4 + 2k^3 + 3k^2 + 2k + 1$

Donc $k^2(k+1)^2 + (k+1)^2 + k^2 = (k^2+k+1)^2$

2° $S = \sum_{k=1}^{n} \sqrt{1 + \dfrac{1}{k^2} + \dfrac{1}{(k+1)^2}}$

$$= \sum_{k=1}^{n} \sqrt{\frac{k^2(k+1)^2 + (k+1)^2 + k^2}{k^2(k+1)^2}}$$
$$= \sum_{k=1}^{n} \sqrt{\frac{(k^2+k+1)^2}{k^2(k+1)^2}}$$
$$= \sum_{k=1}^{n} \frac{k^2+k+1}{k(k+1)} \quad car\ tout\ est\ positif\ sous\ la\ racine$$
$$= \sum_{k=1}^{n} \frac{k(k+1)+1}{k(k+1)}$$
$$= \sum_{k=1}^{n} 1 + \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$
$$= n + \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1}$$
$$= n + \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k}$$
$$= n + 1 - \frac{1}{n+1}$$
$$= \frac{(n+1)^2 - 1}{n+1} = \frac{(n+1-1)(n+1+1)}{n+1}$$
$$= \frac{n(n+2)}{n+1}$$

Exercice 7 :

$$F = \frac{x^2 + 3x + 1}{(x-1)^2(x-2)}$$

Pas de partie entière car $\deg F = -1 < 0$.

$$F = \frac{x^2 + 3x + 1}{(x-1)^2(x-2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}$$

On multiplie par $(x-1)^2$: $\frac{x^2 + 3x + 1}{x-2} = a(x-1) + b + \frac{c(x-1)^2}{x-2}$

$$\frac{1+3+1}{1-2} = b \text{ avec } x = 1$$
$$-5 = b$$

On multiplie par $(x-2)$: $\frac{x^2 + 3x + 1}{(x-1)^2} = \frac{a(x-2)}{x-1} + \frac{b(x-2)}{(x-1)^2} + c$

$$\frac{4+6+1}{(2-1)^2} = c \text{ avec } x = 2$$
$$11 = c$$

$$\frac{0+0+1}{(0-1)^2(0-2)} = \frac{a}{-1} + \frac{-5}{(0-1)^2} + \frac{11}{0-2} \text{ avec } x = 0$$
$$\frac{-1}{2} = -a - 5 - \frac{11}{2}$$
$$a = -5 - \frac{11}{2} + \frac{1}{2} = -10$$

Donc $F = \frac{x^2 + 3x + 1}{(x-1)^2(x-2)} = \frac{-10}{x-1} - \frac{5}{(x-1)^2} + \frac{11}{(x-2)}$

$$G = \frac{4}{(x^2-1)^2}$$

Pas de partie entière car $\deg F = -4 < 0$.

$$G = \frac{4}{(x^2-1)^2} = \frac{4}{((x-1)(x+1))^2} = \frac{4}{(x-1)^2(x+1)^2}$$
$$= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

On multiplie par $(x-1)^2$: $\frac{4}{(x+1)^2} = a(x-1) + b + \left(\frac{c}{x+1} + \frac{d}{(x+1)^2}\right)(x-1)^2$

$$\frac{4}{2^2} = b \text{ avec } x = 1$$
$$1 = b$$

On multiplie par $(x+1)^2$: $\frac{4}{(x-1)^2} = \left(\frac{a}{x-1} + \frac{b}{(x-1)^2}\right)(x+1)^2 + c(x+1) + d$

$$\frac{4}{(-2)^2} = d \text{ avec } x = -1$$
$$1 = d$$

$$\frac{4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{1}{(x-1)^2} + \frac{c}{x+1} + \frac{1}{(x+1)^2}$$
$$\frac{4}{(x-1)^2(x+1)^2} - \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{4 - (x+1)^2 - (x-1)^2}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{4 - (x^2+2x+1) - (x^2-2x+1)}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{4 - x^2 - 2x - 1 - x^2 + 2x - 1}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{2 - 2x^2}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{-2(x-1)(x+1)}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{c}{x+1}$$
$$\frac{-2}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{c}{x+1}$$

On multiplie par $(x-1)$: $\frac{-2}{x+1} = a + \frac{c(x-1)}{x+1}$

$$\frac{-2}{2} = a \text{ avec } x = 1$$
$$-1 = a$$

On multiplie par $(x+1)$: $\frac{-2}{x-1} = \frac{a(x+1)}{x-1} + c$

$$\frac{-2}{-1-1} = c \text{ avec } x = -1$$
$$1 = c$$

Donc $G = \frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$

$$H = \frac{x^5}{x^4 - 1}$$
$$\frac{x^5}{x^4-1} = \frac{x(x^4-1) + x}{x^4-1} = x + \frac{x}{x^4-1}$$

Et $\frac{x}{x^4-1} = \frac{x}{(x^2-1)(x^2+1)}$

$= \frac{x}{(x-1)(x+1)(x^2+1)}$

$= \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$

On multiplie par $(x-1)$: $\frac{x}{(x+1)(x^2+1)} = a + \left(\frac{b}{x+1} + \frac{cx+d}{x^2+1}\right)(x-1)$

$\frac{1}{2\times 2} = a$ avec $x = 1$

$\frac{1}{4} = a$

On multiplie par $(x+1)$: $\frac{x}{(x-1)(x^2+1)} = \frac{a(x+1)}{x-1} + b + \frac{(cx+d)(x+1)}{x^2+1}$

$\frac{-1}{-2\times 2} = b$ avec $x = -1$

$\frac{1}{4} = b$

On a donc $\frac{x}{(x-1)(x+1)(x^2+1)} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x+1} + \frac{cx+d}{x^2+1}$

$\frac{x}{(x-1)(x+1)(x^2+1)} - \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} = \frac{cx+d}{x^2+1}$

$\frac{4x-(x+1)(x^2+1)-(x-1)(x^2+1)}{4(x-1)(x+1)(x^2+1)} = \frac{cx+d}{x^2+1}$

$\frac{4x-(x^2+1)(x+1+x-1)}{4(x-1)(x+1)(x^2+1)} = \frac{cx+d}{x^2+1}$

$\frac{2x(2-x^2-1))}{4(x-1)(x+1)(x^2+1)} = \frac{cx+d}{x^2+1}$

$\frac{-2x(x^2-1)}{4(x-1)(x+1)(x^2+1)} = \frac{cx+d}{x^2+1}$

$\frac{-\frac{1}{2}x}{x^2+1} = \frac{cx+d}{x^2+1}$

Par identification, on obtient $c = \frac{-1}{2}$ et $d = 0$.

⇨ On pouvait « tricher » et multiplier par (x^2+1) et remplacer x par i… 😊

Donc $H = x + \frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)}$

2. $I = \frac{1}{x(x+1)} = \frac{x+1-x}{x(x+1)}$

$= \frac{x+1}{x(x+1)} - \frac{x}{(x(x+1))}$

$= \frac{1}{x} - \frac{1}{x+1}$

$J = \frac{1}{x^3(x^3+1)} = \frac{1}{x^3} - \frac{1}{x^3+1}$ d'après I

$= \frac{1}{x^3} - \frac{1}{(x+1)(x^2-x+1)}$

On a $\frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$

On multiplie par $(x+1)$: $\frac{1}{x^2-x+1} = a + \frac{(bx+c)(x+1)}{x^2-x+1}$

$\frac{1}{1+1+1} = a$ et donc $a = \frac{1}{3}$

On a donc $\frac{1}{(x+1)(x^2-x+1)} = \frac{\frac{1}{3}}{x+1} + \frac{bx+c}{x^2-x+1}$

$\frac{1}{(x+1)(x^2-x+1)} - \frac{1}{3}\times\frac{1}{x+1} = \frac{bx+c}{x^2-x+1}$

$\frac{3-(x^2-x+1)}{3(x+1)(x^2-x+1)} = \frac{bx+c}{x^2-x+1}$

$\frac{3-x^2+x-1}{3(x+1)(x^2-x+1)} = \frac{bx+c}{x^2-x+1}$

$\frac{-x^2+x+2}{3(x+1)(x^2-x+1)} = \frac{bx+c}{x^2-x+1}$

$\frac{-(x+1)(x-2)}{3(x+1)(x^2-x+1)} = \frac{bx+c}{x^2-x+1}$

$\frac{-x+2}{3(x^2-x+1)} = \frac{bx+c}{x^2-x+1}$

$-\frac{1}{3}x + \frac{2}{3} = bx + c$

Par identification, on obtient $b = \frac{-1}{3}$ et $c = \frac{2}{3}$.

Donc $J = \frac{1}{x^3} - \left(\frac{1}{3(x+1)} + \frac{-1x+2}{3(x^2-x+1)}\right) = \frac{1}{x^3} - \frac{1}{3(x+1)} + \frac{x-2}{3(x^2-x+1)}$

<u>Exercice 8</u> :

$S_1 = \sum_{k=1}^{n} \frac{2}{k^2+2k}$

On a $\frac{2}{k^2+2k} = \frac{2}{k(k+2)} = \frac{a}{k} + \frac{b}{k+2}$

On multiplie par k : $\frac{2}{k+2} = a + \frac{bk}{k+2}$

$\frac{2}{0+2} = a$ et donc $1 = a$

On multiplie par $(k+2)$: $\frac{2}{k} = \frac{a(k+2)}{k} + b$

$\frac{2}{-2} = b$ et donc $-1 = b$

$$S_1 = \sum_{k=1}^{n} \frac{2}{k^2+2k}$$
$$= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2}$$
$$= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+2}$$
$$= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=3}^{n+2} \frac{1}{k}$$
$$= 1 + \frac{1}{2} + \sum_{k=3}^{n} \frac{1}{k} - \left(\sum_{k=3}^{n} \frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2} \right)$$
$$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_2 = \sum_{k=2}^{n} \frac{1}{k^3-k}$$

On a $\frac{1}{k^3-k} = \frac{1}{k(k^2-1)} = \frac{1}{k(k-1)(k+1)} = \frac{a}{k} + \frac{b}{k-1} + \frac{c}{k+1}$

On multiplie par k : $\frac{1}{(k-1)(k+1)} = a + \frac{bk}{k-1} + \frac{ck}{k+1}$

$\frac{1}{-1 \times 1} = a$ et donc $-1 = a$

On multiplie par $(k-1)$: $\frac{1}{k(k+1)} = \frac{a(k-1)}{k} + b + \frac{c(k-1)}{k+1}$

$\frac{1}{1(1+1)} = b$ et donc $\frac{1}{2} = b$

On multiplie par $(k+1)$: $\frac{1}{k(k-1)} = \frac{a(k+1)}{k} + \frac{b(k+1)}{k-1} + c$

$\frac{1}{-1(-1-1)} = c$ et donc $\frac{1}{2} = c$

Donc $\frac{1}{k^3-k} = \frac{-1}{k} + \frac{\frac{1}{2}}{k-1} + \frac{\frac{1}{2}}{k+1}$

D'où $S_2 = \sum_{k=2}^{n} \frac{-1}{k} + \frac{\frac{1}{2}}{k-1} + \frac{\frac{1}{2}}{k+1}$

$$= -\sum_{k=2}^{n} \frac{1}{k} + \frac{1}{2}\sum_{k=2}^{n} \frac{1}{k-1} + \frac{1}{2}\sum_{k=2}^{n} \frac{1}{k+1}$$
$$= -\sum_{k=2}^{n} \frac{1}{k} + \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{2}\sum_{k=3}^{n+1} \frac{1}{k}$$
$$= -\sum_{k=2}^{n} \frac{1}{k} + \frac{1}{2}\left(1 + \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{n}\right) + \frac{1}{2}\left(\sum_{k=2}^{n} \frac{1}{k} - \frac{1}{2} + \frac{1}{n+1}\right)$$
$$= \frac{1}{2} - \frac{1}{2n} - \frac{1}{4} + \frac{1}{2(n+1)}$$
$$= \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$$
$$= \frac{n(n+1) - 2(n+1) + 2n}{4n(n+1)}$$
$$= \frac{n^2+n-2n-2+2n}{4n(n+1)}$$
$$= \frac{n^2+n-2}{4n(n+1)}$$
$$= \frac{(n-1)(n+2)}{4n(n+1)}$$

Exercice 9 :

1° $u_n = \ln \frac{n^3}{(n+2)(n-1)^2}$
$= 3\ln n - (\ln(n+2) + \ln(n-1)^2)$
$= 3\ln n - \ln(n+2) - 2\ln(n-1)$

$$\sum_{k=2}^{n} u_k = \sum_{k=2}^{n} 3\ln k - \ln(k+2) - 2\ln(k-1)$$
$$= 3\sum_{k=2}^{n} \ln k - \sum_{k=2}^{n} \ln(k+2) - 2\sum_{k=2}^{n} \ln(k-1)$$
$$= 3\sum_{k=2}^{n} \ln k - \sum_{k=4}^{n+2} \ln k - 2\sum_{k=1}^{n-1} \ln k$$
$$= 3\sum_{k=2}^{n} \ln k - \left(\sum_{k=2}^{n} \ln k - \ln 2 - \ln 3 + \ln(n+1) + \ln(n+2) \right) - 2\left(\sum_{k=2}^{n} \ln k + \ln 1 - \ln n \right)$$
$$= \ln 2 + \ln 3 - \ln(n+1) - \ln(n+2) - 2\ln 1 + 2\ln n$$
$$= \ln 6 - \ln(n+1) - \ln(n+2) + \ln n^2$$
$$= \ln \frac{6n^2}{(n+1)(n+2)}$$

2° $\lim_{n\to+\infty} \frac{6n^2}{(n+1)(n+2)} = \lim_{n\to+\infty} \frac{6n^2}{n^2+3n+2}$
$= \lim_{n\to+\infty} \frac{6n^2}{n^2}$ *monômes de plus hauts degrés*
$= 6$

D'où $\lim_{n\to+\infty} \sum_{k=2}^{n} u_k = \ln 6$

La série converge donc vers $\ln 6$